ICARUS 11, 95–103 (1969)

Resonances in Complex Systems: A Reply to Critiques

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Communicated by B. Y. Levin

Received August 19, 1968; revised May 23, 1969

The conclusions reached by Backus and Hénon, that resonance relations in the solar system of the sort proposed by Molchanov are a result of chance, is based on a very crude statistical model. A more accurate model gives a value $P \sim 10^{-10}$ for the probability of chance formation of systems similar to the solar system.

I. Introduction

The present paper is a reply to the critiques by Backus (1969) and Hénon (1969) of my paper (Molchanov, 1968) arguing for resonances in the periods of the planets. [See also Dermott, (1969).] In my opinion the statistical model employed by Backus (1969) is unsatisfactory on four counts:

(1) The Euclidean metric postulated is not characteristic of the frequency space.

(2) The significance of nearby planets compared with major and especially distant planets is exaggerated.

(3) The definition of good systems is inadequate. Their number is sharply overstated.

(4) The structure of the solar system has been oversimplified. Among the four basic subsystems only planets have been considered.

A better model, free of these shortcomings, was presented in the spring of 1967 in Moscow at a conference on celestial mechanics. A translation of that paper follows in this same issue of *Icarus* (Molchanov, 1969). As an alternative model is being presented it is necessary to analyze the underlying premises of each model.

II. Proximity in Frequency Space

Backus uses formulas from analytical geometry to measure the proximity of frequency vectors. This is equivalent to postulating Euclidean metrics in frequency space. The meaning of such a postulate is difficult to understand, and it is doubtful whether such a simple meaning can be given to motion in frequency space.

Frequency intervals have been measured for a long time in the theory of oscillations, i.e., musical acoustics. The tempered system which is commonly used is a logarithmically isometric scale in which the octave (ratio of frequencies 1:2) is divided into 12 equal intervals (semitones). It follows that frequency intervals are measured by the logarithms of the ratio of the frequencies.

The important question about the closeness of a tempered system to a pure system serves as a good indicator to the problem of the reality of resonances in the solar system; this has already been resolved in musical acoustics. Thus the "well-tempered" clavier should be likened to an experimentally observed system; the pure system

$$\{1, 8/9, 4/5, 3/4, 2/3, 8/15, 1/2\} \qquad (1)$$

can be considered as a set of exactly resonant frequencies which approximate the tempered system given by

$$2^{-0/12}, 2^{-2/12}, 2^{-4/12}, 2^{-5/12}, 2^{-7/12}, 2^{-9/12}, 2^{-11/12}, 2^{-12/12}. \qquad (2)$$

These vectors will be used to construct a table analogous to the tables for the subsystems of the solar system. The measure of closeness $\ln[\omega_{obs}/\omega_{theor}]$ can be

TABLE I

RESONANCE VECTORS AND FREQUENCIES OF A WELL-TEMPERED SCALE

Interval ("Planet")	Clavier ("obs")	Pure system ("theor")	$\Delta\omega/\omega$								
Prima	1.0000	1.0000	—	0	0	−2	0	0	0	3	0
Second	0.8909	0.8889	0.0022	0	0	3	0	0	−4	0	0
Third	0.7937	0.8000	−0.0079	−2	0	0	0	3	0	0	0
Fourth	0.7492	0.7500	−0.0011	1	0	0	0	0	0	0	−2
Fifth	0.6674	0.6667	0.0011	0	3	0	0	−4	0	0	0
Sixth	0.5946	0.6000	−0.0090	0	0	0	−2	0	0	0	3
Seventh	0.5297	0.5333	−0.0068	−4	0	5	0	0	0	0	0
Octave	0.5000	0.5555	0.0000	—	—	—	—	—	—	—	—

replaced in all cases by the simpler expression $\Delta\omega/\omega$. In fact

$$\ln\frac{\omega_{obs}}{\omega_{theor}} = \ln\frac{\omega+\Delta\omega}{\omega} = \ln\left(1+\frac{\Delta\omega}{\omega}\right) = \frac{\Delta\omega}{\omega} - \frac{1}{2}\left(\frac{\Delta\omega}{\omega}\right)^2 + \cdots \approx \frac{\Delta\omega}{\omega}. \quad (3)$$

The value in the $\Delta\omega/\omega$ column is of the same order as for the solar system. Therefore, Backus's assertion would result in mass unemployment among those who tune musical instruments. Why tune a violin (the harp and pianoforte are discussed below) if any random set of strings will sound 45 times purer than a well-tempered scale [see Eq. (2), Backus (1969)]?

This mistaken argument by *reductio ad absurdum* presents a serious question about the correct definition of the class of good systems. A detailed treatment of the question is given below.

III. The Role of Distant Planets

When calculating the discrepancy using Eq. (5) (Backus, 1969) it is found that Mercury's role is 50 times greater than that of Jupiter and 1000 times greater than that of Pluto. In other words, an error of 0.1% in Mercury's frequency gives an error of 5% in Jupiter's frequency; Pluto's frequency will not even be calculable. It is strange that Mercury becomes so important compared with Jupiter although they are only distantly related and despite what was said in Section II. This suggests that the use of Euclidean metrics is inappropriate in frequency space. An analogy with musical instruments is useful at this point. If all the strings of a pianoforte are well tuned except one (which, say, is broken) Eq. (5) will not distinguish this instrument from a group of strings tightened at random.

IV. Good Systems

The essential difference between the two rival models lies in the definition of the class of "good" systems. A definition of good resonance is given by Backus and it is stated that a more restricted class of system is "difficult to imagine." In fact this is not only possible, but is precisely what must be done; it is dealt with in the original work on the subject by Molchanov (1968).

Backus has adopted an approach which is methodologically unsatisfactory in that resonances are examined individually and independent of one another. Thus the system as a whole could appear bad if individual resonances are bad as described in Section VI, below. In addition to the composite parts of a system being good, it is necessary that their combination also be good. This condition of mutual compatibility decreases the number of "good" systems.

An alternative definition of the class of good systems, discussed in detail in the following paper, is obtained from the physical meaning of resonance vectors. The elements of such a vector are coefficients of the resonant phase, i.e., the first integral of the unperturbed system. But *any* linear combination of resonant phases is also a resonant phase. Thus, various sets of resonant phases can produce one and the same system. [Because of this Backus's estimate of the number of good systems is too high. Our calculation results in an overestimation too, but it is not so excessive as to distort the entire picture.] However, all the available methods are not equally valid. A one-to-one substitution of phase variables is not permissible with all sets of resonant phases. For example, suppose

$$\tilde{\Psi}_1 = \phi_1 - 2\phi_2$$
$$\tilde{\Psi}_2 = \phi_1 - 2\phi_3.$$

Replacing these phases by an equivalent system, we have

$$\tilde{\tilde{\Psi}}_1 = \tilde{\Psi}_1 = \phi_1 - 2\phi_2$$
$$\tilde{\tilde{\Psi}}_2 = \tilde{\Psi}_1 - \tilde{\Psi}_2 = 2\phi_3 - 2\phi_2.$$

We see that the multiple phase $\tilde{\tilde{\Psi}}_2$ is concealed in the original system. Reducing this to an equivalent pair we obtain the phases

$$\Psi_1 = \phi_1 - 2\phi_2$$
$$\Psi_2 = \phi_2 - \phi_3.$$

This permits extension to a system of phases. One-third of the phase (nonresonant) can be chosen by different means. Thus, for example,

$$\Psi_3 = \phi_1 - \phi_2.$$

Naturally in the multiple-frequency situation multiple phases are more likely to be concealed and their recovery is not a simple matter. A general method is given by Molchanov (1966) and is based on the following theorem:

THEOREM I

An integral (not necessarily square) matrix N can be represented as a product $N = TA$ of the triangular integral matrix T and the square integral matrix A with determinant equal to 1.

In the triangular matrix T the elements above the diagonals are zeros. It may have an incomplete number of rows (equal to the number of resonance vectors, i.e., the rows in matrix N) and in this sense T is a right triangle; the nonzero elements of this matrix lie in the right triangle. Nonzero diagonal elements of T indicate the presence of multiple phases in the system N. The upper rows of matrix A (which are equal in number to the rows in matrix N) form a resonant system equivalent to the system N, but permitting addition to the whole matrix of the change of phase variables.

The rarity of systems like the solar system has been evaluated in the following paper and is based on the properties of matrix A. In my opinion the matrices N used by Backus cannot give a correct picture. The example given above where

$$N = \begin{pmatrix} 1 & -2 & 0 \\ 1 & 0 & -2 \end{pmatrix}, \quad A = \begin{pmatrix} 1 & -2 & 0 \\ 0 & 1 & -1 \\ 1 & -1 & 0 \end{pmatrix},$$
$$T = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \end{pmatrix}$$

shows that the matrices are analogous to reducible fractions; their number is much greater than the number of irreducible fractions, to which matrices A are analogous.

V. Satellite Systems

The arguments presented above are not a strict proof in themselves. A proof can only be contained in the formulation of an exact theory of resonance states within the framework of complete systems of equations with all perturbations taken into account. In particular, the question of stability must be resolved. Why are planets and satellites locked into simple resonances whereas the rings of Saturn and the asteroid belt have gaps in these places? Also, why are resonances of axial rotation treated as stable and well known [1:1, "prima," for the Moon and 2:3, a fifth, for Mercury]? Does this not mean that Mercury is more likely a satellite of the Sun than the common planet? These questions cannot be answered in the framework of an unperturbed system, and remain for future solution.

If the arguments in favor of the reality of resonances do not appear convincing, then in the author's opinion the arguments against them are even less persuasive. In particular, the studies should not be restricted to planets alone. The principle of maximum resonance is formulated for any system which has attained evolutionary maturity; and the unique structure of satellite systems is important because it confirms the generality of this principle.

The probability of only planets being close to resonance is given in the following paper by

$$P < 10^{-4}.$$

If we consider the probability of coincidence of the four subsystems (the planets and the moons of Jupiter, Saturn, and Uranus) as a result of chance (i.e., closeness of each of the four subsystems to resonance) then P is given more closely by

$$P < 10^{-10}.$$

The discrepancy always has the same sign for satellite systems—negative for Uranus and positive for Jupiter and Saturn (Molchanov, 1969). Therefore the proximity can be improved by choosing the frequency scale in another manner. This is especially evident for Saturn where simply the selection of Dione as the calibrating body instead of Titan noticeably decreases the discrepancy $\Delta\omega/\omega$.

VI. Is the Well-Tempered Clavier a Good System?

When dealing with problems on the reality of resonances it is very important to understand what is meant by a good system. The difference between the two alternative definitions is useful to illustrate a theoretically interesting example of a pure system. For simplicity we shall denote the system by K; its resonance matrix N is of the form

$$N = \begin{Vmatrix} 0 & 0 & -2 & 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 3 & 0 & 0 & -4 & 0 & 0 \\ -2 & 0 & 0 & 0 & 3 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & -2 \\ 0 & 3 & 0 & 0 & -4 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 & 0 & 0 & 0 & 3 \\ -4 & 0 & 5 & 0 & 0 & 0 & 0 & 0 \end{Vmatrix}.$$

All the rows of this system belong to the class, K, of all rational hyperplanes $\mathbf{n}$ for which $n_2 = 0$ in at least six cases out of eight and $|n_i| \leqslant 5$ in the remaining two cases. In musical acoustics this is the class of frequency intervals containing only primes, thirds, fourths, and fifths with octavial rises; it does not contain seconds.

This class is really quite similar to Backus's class (i). It contains 3388 elements in all instead of the 28,3500 elements in class (i):

$$\frac{8!}{6!\,2!} \times 11^2 = 3388.$$

Nevertheless K is not good in the same sense as are good subsystems of the solar system. To make certain of this it is necessary to find a matrix A whose construction will also determine whether or not it belongs to good systems. Therefore matrix N must be reduced to a triangular matrix T. At the same time we shall ascertain that multiple phases are concealed in matrix N.

The method used to find A is based on the simultaneous construction of rows of matrix A and columns of the inverse matrix $B = A^{-1}$. These rows $\mathbf{a}_i$ and $\mathbf{b}_k$ are biorthogonal as shown by the equality

$$AB = E$$

(where E is the unit matrix) and the rule for matrix multiplication, i.e., rows from the left and columns from the right. Assume (induction from the number of rows in matrix N) that we have already constructed S rows and columns $\mathbf{a}_i$ and $\mathbf{b}_k$ so that the following relations are satisfied:

$$\begin{aligned} \mathbf{n}_1 &= T_{11}\mathbf{a}_1 \\ \mathbf{n}_2 &= T_{21}\mathbf{a}_1 + T_{22}\mathbf{a}_2 \\ \vdots\ & \quad \vdots \\ \mathbf{n}_s &= T_{s1}\mathbf{a}_1 + T_{s2}\mathbf{a}_2 + \cdots T_{ss}\mathbf{a}_s. \end{aligned}$$

Examine the next row $\mathbf{n}_{s+1}$ and we see that the vector $\mathbf{m}_{s+1}$ is orthogonal to all the preceding $\mathbf{b}_i s$:

$$\mathbf{m}_{s+1} = \mathbf{n}_{s+1} - T_{s+1,1}\mathbf{a}_1 \cdots T_{s+1,s}\mathbf{a}_s,$$

where the numbers $T_{s+1,i}$ are found from the orthogonality conditions

$$T_{s+1,i} = (\mathbf{n}_{s+1}, \mathbf{a}_i).$$

The following lemma contains the most important difference between the integral case and the usual method of orthogonalization:

LEMMA

For an integral vector (rows) $\mathbf{m}$, an integral vector (columns) $\mathbf{c}$ will be found so that their scalar product is equal to the greatest common divisor of the elements of vector $\mathbf{m}$:

$$d = (\mathbf{m}, \mathbf{c}).$$

Result

Any vector $\mathbf{m}$ is proportional to a vector $\mathbf{a}$ which has an adjoint column $\mathbf{c}$

$$\mathbf{m} = d\mathbf{a}, \qquad (\mathbf{a}, \mathbf{c}) = 1.$$

The Euclidean algorithm for searching for the greatest common divisor of two numbers gives proof of the Lemma on a plane. The generalization is obtained by induction.

If the rows of matrix N are linearly independent then the vector $\mathbf{m}_{s+1}$ is different from zero. Applying the Lemma above shows concealed multiple phases and gives the number $T_{s+1,s+1}$ (from which we can also obtain the multiplicity factor of the phase $\mathbf{m}_{s+1}$) and column $\mathbf{c}_{s+1}$; from this

$$\mathbf{m}_{s+1} = T_{s+1,s+1}\,\mathbf{a}_{s+1}, \qquad (\mathbf{a}_{s+1}, \mathbf{c}_{s+1}) = 1.$$

We have already obtained the general row for matrices A and T as

$$\mathbf{a}_{s+1} = \text{row}_{s+1} A,$$
$$\mathbf{n}_{s+1} = T_{s+1,1}\,\mathbf{a}_1 + \cdots + T_{s+1,s}\,\mathbf{a}_s + T_{s+1,s+1}\,\mathbf{a}_{s+1}.$$

However, $\mathbf{c}_{s+1}$ cannot yet be written in the $s+1$ column of matrix B. It must first be orthogonalized to all the preceding $\mathbf{a}_i s$. This is analogous to the procedure for correcting $\mathbf{n}_{s+1}$:

$$\mathbf{b}_{s+1} = \mathbf{c}_{s+1} - Q_{1,s+1}\,\mathbf{b}_1 \cdots Q_{s,s+1}\,\mathbf{b}_s,$$

where

$$Q_{i\,s+i} = (\mathbf{a}_i, \mathbf{c}_{s+1}).$$

From the orthogonality obtained earlier of $\mathbf{a}_{s+1}$ to all preceding $\mathbf{b}_i$, $\mathbf{b}_{s+1}$ remains adjoint to $\mathbf{a}_{s+1}$:

$$(\mathbf{a}_{s+1}, \mathbf{b}_{s+1}) = (\mathbf{a}_{s+1}, \mathbf{c}_{s+1}) = 1.$$

Remarks

If matrix N has an incomplete number of rows as in the resonance matrix, then to obtain a square matrix it is sufficient to add missing rows, arbitrarily, provided that they are linearly independent. A row such as $\mathbf{n}_8$ can be added to matrix N; in particular

$$\mathbf{n}_8 = (0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 1).$$

By applying this method to matrix N we do not obtain the multiple phase in the first six steps. Because of this the triangular matrix T (of the incomplete, sixth order) has unities on the main diagonal and may be included in matrix A. This is an important point—different matrices may determine one and the same system. When they are selected arbitrarily the estimated number of good systems is much too high. In fact there are even fewer good systems than our calculations would indicate. Backus's estimate is so exaggerated that it obscures the true situation.

Returning to the example, let us discuss in greater detail the nascent state of the multiple phase. After six steps, the following situation arises: the resonance matrix is

$$A = \begin{Vmatrix} 0 & 0 & -2 & 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 3 & 0 & 0 & -4 & 0 & 0 \\ -2 & 0 & 0 & 0 & 3 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & -2 \\ 0 & 3 & 0 & 0 & -4 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 & 0 & 0 & 0 & 3 \end{Vmatrix}$$

and from its first six rows the matrix of six biorthogonal columns is constructed:

$$B = \begin{Vmatrix} 0 & 0 & 4 & 9 & 12 & 18 \\ 0 & 0 & 4 & 8 & 11 & 16 \\ 4 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 6 & 9 & 13 \\ 0 & 0 & 3 & 6 & 8 & 12 \\ 3 & 2 & 0 & 0 & 0 & 0 \\ 3 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 4 & 6 & 9 \end{Vmatrix}.$$

Now consider the seventh row

$$\mathbf{n}_7 = (-4 \quad 0 \quad 5 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0)$$

and from it subtract the components along the six preceding rows:

$$\mathbf{m}_7 = \mathbf{n}_7 - T_{71}\mathbf{a}_1 \cdots - T_{76}\mathbf{a}_6.$$

In appearance the seventh row is not very different from the six preceding rows. The two nonzero elements in it are not very large and it would seem that a multiple phase could not be concealed anywhere in such a row. Nevertheless there is a dodecatuple phase hidden in it.

This is an example of detailed computation and it demonstrates the difference between resonance considered individually, and the same resonance as part of a system of resonances.

Calculating the coefficients

$$T_{7K} = (\mathbf{n}_7, \mathbf{b}_K)$$

we obtain

$$T_{71} = 20, \qquad T_{72} = 15, \qquad T_{73} = -16,$$
$$T_{74} = -36, \qquad T_{75} = -48, \qquad T_{76} = -72.$$

Thus

$$\mathbf{m}_7 = (0 \quad 144 \quad 0 \quad -144 \quad -144 \quad 60 \quad -60 \quad 144).$$

Consequently now $\mathbf{n}_7$ in the system $\mathbf{a}_1$, $\mathbf{a}_6$ is equivalent to the multiple row $\mathbf{m}_7$ and does not permit expansion to the unimodular matrix A. To make this possible it is necessary to divide the multiple phase by the greatest common divisor of its coefficients and

$$\mathbf{t}_7 = (0 \quad 12 \quad 0 \quad -12 \quad -12 \quad 5 \quad -5 \quad 12).$$

The new row is equivalent (in the system of the first six rows) to both rivals, but can be included in a matrix with a determinant of unity. The construction is still incomplete. The large coefficients of the new row have to be decreased by deducting from it the integral combination of preceding rows. The equivalence and nonmodularity remain conserved. Evidently, the best combination to use would be

$$\mathbf{a}_7 = \mathbf{t}_7 + \mathbf{a}_1 + \mathbf{a}_2 - 3\mathbf{a}_5 - 4\mathbf{a}_6$$

with small elements

$$\mathbf{a}_7 = (0 \quad 3 \quad 1 \quad -4 \quad 0 \quad 1 \quad -2 \quad 0).$$

When rows are changed, columns must also be changed so that the inverse transformation can be accomplished. This is easy to illustrate by the formula

$$(TA)(BT^{-1}) = T(AB)T^{-1} = TT^{-1} = E.$$

After all these manipulations the "elegant" matrix A which contains all the resonances in the first seven rows and with determinant one is given by

$$A = \begin{Vmatrix} 0 & 0 & 2 & 0 & 0 & 0 & -3 & 0 \\ 0 & 0 & -3 & 0 & 0 & 4 & 0 & 0 \\ -2 & 0 & 0 & 0 & 3 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & -2 \\ 0 & 3 & 0 & 0 & -4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 & 0 & -3 \\ 0 & 3 & 1 & -4 & 0 & 1 & -2 & 0 \\ 0 & -2 & 0 & 1 & 1 & -2 & 2 & 1 \end{Vmatrix}.$$

From the inverse $B = A^{-1}$

$$B = \begin{Vmatrix} 72 & 72 & 4 & 9 & 48 & 54 & 72 & 180 \\ 64 & 64 & 4 & 8 & 43 & 48 & 64 & 160 \\ 56 & 57 & 0 & 0 & 36 & 48 & 60 & 144 \\ 54 & 54 & 3 & 6 & 36 & 41 & 54 & 135 \\ 48 & 48 & 3 & 6 & 32 & 36 & 48 & 120 \\ 42 & 43 & 0 & 0 & 27 & 36 & 45 & 108 \\ 37 & 38 & 0 & 0 & 24 & 32 & 40 & 96 \\ 36 & 36 & 2 & 4 & 24 & 27 & 36 & 90 \end{Vmatrix}.$$

It is instructive to compare the structural matrix of a well-tempered clavier with matrices of planets and satellites; the difference in the value of the coefficients is expected. In matrices of the solar system (Molchanov, 1969) over half the nonzero elements are unity—45 out of 77. In the present case nonzero coefficients equal to unity form less than a third of the total—6 out of 23.

However, the main difference is in the structure of the matrices. In the clavier matrix nonzero elements are distributed throughout the square field of size $8 \times 8 = 64$. Also matrices of the solar system are almost triangular. The following simple calculation illustrates the difference qualitatively.

Consider the number of ways in which the 23 nonzero elements can be located.

In clavier matrices this number of combinations is given by

$$C_{64}^{23} = \frac{64!}{23!\,41!}.$$

In nearly triangular matrices of the solar system 5 nonzero elements should be placed on the diagonal. The remaining 18 positions can be chosen arbitrarily from the 28 positions above the diagonal and 3 below it in the lower right-hand corner. In all,

$$C_{31}^{18} = \frac{31!}{18!\,13!}.$$

The ratio of these numbers can be obtained from Stirling's formula and it is really quite large:

$$\log[C_{64}^{23}/C_{31}^{18}] \simeq 8.9.$$

Thus triangularity alone, without consideration of the arrangement of nonzero elements near the diagonal, leads to a billionfold difference in the complete set of matrices. Structural matrices of the solar system are better in the sense that they belong to a more restricted class than the structural matrix of a well-tempered clavier.

VII. Hénon's Resonances

It appears that the critical comments made by Hénon (1969) are a result of some misunderstandings. He feels that "good" frequencies are situated on the axis of frequencies with a density $8/R$ where $R = 4536$ times the number of "good" resonances

$$n\omega_i \pm \omega_j \pm \omega_k = 0.$$

This would have been true if all the frequencies lay on a segment of length unity. In fact, Hénon's resonances are worse than that of a well-tempered clavier. There is a considerable amount of scatter in his resonances. For example, suppose $n = 4$ in the first four equations (4 being the intermediate value between 2 and 7):

$$\begin{aligned} \omega_1 - 4\omega_2 &= 0 \\ \omega_2 - 4\omega_3 &= 0 \\ \omega_3 - 4\omega_4 &= 0 \\ \omega_4 - 4\omega_5 &= 0. \end{aligned}$$

If $\omega_5 = 1$ then $\omega_1 = 256$ and the expected density of good points will be $\Delta\omega \simeq 0.46$ and not $\Delta\omega \simeq 0.0018$ as proposed by Hénon.

There is another fact relating to Hénon's artificial system. It is a "good" system. With 10^{19} good systems by my definition (Molchanov, 1969), in one of the 10^{10} neighboring galaxies somewhere there is precisely such a planetary system; but do the frequencies of our solar system serve to refute this other system's resonant structure?

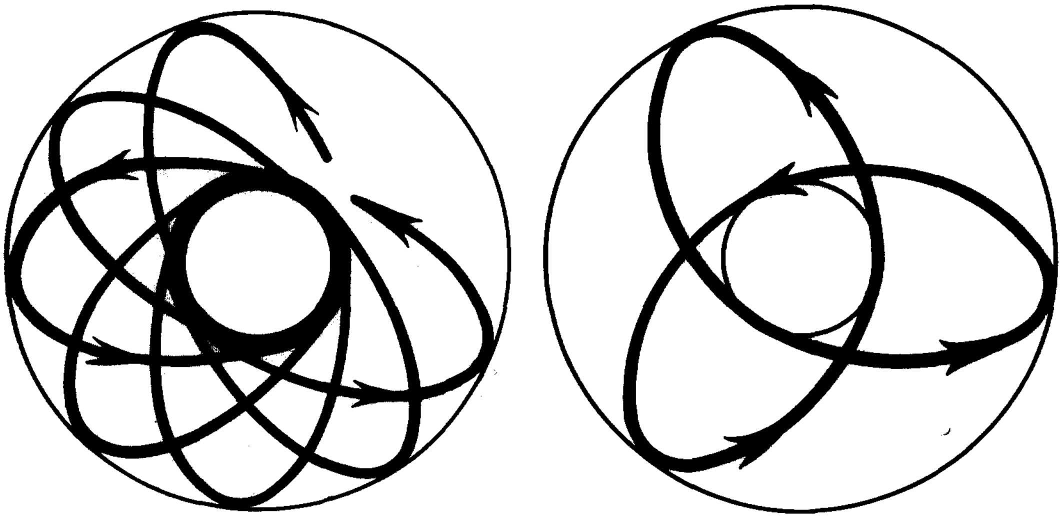
For a mathematically correct estimate of the probability, P, N trials are necessary, where $N \gg N_0 = 1/P$. The following paper shows that $N_0 > 10^4$. As far as random numbers relate to solving this problem Hénon says "It is well known, however, that intuition can be very misleading in matters of probability. Chance alone can produce seemingly highly improbable results." I agree. But artificial systems can be such highly improbable results.

I would venture to suggest that chance has profited by circumstances to cause damage to the domain of determinism.

VIII. General Observations

I am grateful to the critics who have pointed out, quite convincingly, the weakness in the principle of resonance—namely the lack of a definition of good systems—in the form in which it was presented before 1967. This confirms the timeliness with which the definition was made more precise (1967), and as we have attempted to show, such a definition is necessary to nullify the critical remarks made by Backus and Hénon. But a precise definition by itself is insufficient and inconclusive. As long as the heterogeneity of the planets has not been taken into account, we cannot consider that the structural principles of the solar system are thoroughly understood.

It is possible that heterogeneity is generally characteristic of multiple-frequency systems although it does vary quite widely in the way it manifests itself. Historically the resonance phenomenon was first discovered in music, i.e., in an essentially biological system (It served



FIGS. 1 and 2. 1. Two-frequency motion. Incommensurable. 2. Resonance 2:3 in the general case.

as the reason for the founding of an entire philosophical school—the Pythagorean mystics of whole numbers), and only much later appeared in quantum systems. Therefore, good resonant systems cannot be the same in all cases. In acoustics and quantum physics the structure of a given system can be judged from its frequency spectrum; in mechanics this still remains an open question (see Section VI). The spectrum of a planetary system, like that of many quantum systems, is strongly rarefied—9 frequencies in 10 octaves—while in acoustics there are 12 strings in one octave; the whole visible optical spectrum is contained in a seventh.[1] Therefore it is methodologically interesting to study "intermediate" mechanical systems such as asteroids, the rings of Saturn or resonances of rotation with revolution (see Jeffreys, 1969).

[1] Why one octave is sufficient in optics while 9 octaves are inadequate in acoustics, is an interesting biological question.

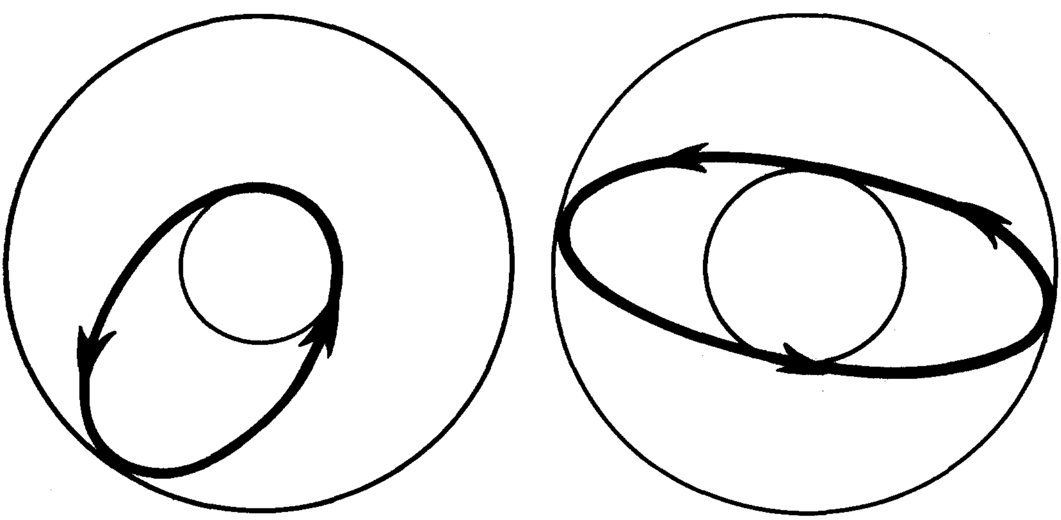


FIG. 3. Identical resonance; 1:1 of a coulomb field.

FIG. 4. Identical resonance 1:2 of a harmonic oscillator.

In conclusion, once again (Molchanov, 1966) we note that the rule of maximum resonance is applicable even to the motion of one planet.

It is well-known (Landau and Lifshits, 1958) that motion in the central potential field $U(r)$ is always horizontal, generally double-frequencied, and fills up a whole ring. The angular frequency ω_ϕ is related to the radial frequency ω_r by

$$\omega_\phi - k\omega_r = 0.$$

The coefficient of ω_r depends on the energy E, momentum M and mass m of the particle

$$k = k(E, M, m) = \frac{1}{\pi}\int_{r_{min}}^{r_{max}} \frac{(M/r^2)\,dr}{\{2m[E - U(r)] - M^2/r^2\}^{1/2}},$$

where the limits of integration r_{min} and r_{max} are the essential classical turning points, i.e., the solutions of the equation

$$2m[E - U(r)] - (M^2/r^2) = 0.$$

In any field $U(r)$ there exist resonant values of energy and momentum for which the frequencies become commensurable,

$$k = p/q.$$

There are only two potentials for which the resonance is identical for all values of momentum and energy. The coulomb and, in particular, the Newtonian potential correspond to 1 : 1 resonance (unison); 1 : 2 resonance (octave) corresponds to the potential of a harmonic oscillator (see Figs. 1–4).

It is of interest that dipole moments—the foundation of chemistry—can exist only in a coulomb field. If the coulomb field vanishes, no other field can reassemble atoms into molecules. The author is grateful to his friend, Dr. E. E. Shnoll, who brought this extraordinary fact to his attention.

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